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Simultaneous Rational Approximants for a Pair of Functions with Smooth Maclaurin Series Coefficients

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Let $f(z)=\sum_{j=0}^{\infty} a_j z^j$ and $g(z)=\sum_{j=0}^{\infty} b_j z^j$ be formal power series for which the quantities $a_{j+1}a_{j-1}/a_j^2$ and $(b_j/b_{j+1})/(a_j/a_{j+1})$ have a prescribed asymptotic behaviour as $j\to\infty$. We obtain the asymptotic behaviour as $l\to\infty$ of the $(l-s, r, s)$, $l, r, s\in\mathbb{N}$, Hermite–Padé approximant to (f, g) and the associated determinants. © 1995 Academic Press, Inc.

1. INTRODUCTION

Given a vector $(f_1, ..., f_m)$ of formal power series, a vector of rational functions $(Q_{1n}/Q_n, ..., Q_{mn}/Q_n)$ can be defined in a natural way such that each component Q_{jn}/Q_n interpolates f_j at the origin with a degree as high as possible. All components in the vector $(Q_{1n}/Q_n, ..., Q_{mn}/Q_n)$ have a common denominator polynomial Q_n. Although the definition is straightforward, it is difficult to make any meaningful statements about properties like uniqueness or convergence of these simultaneous rational approximants without more special assumptions. In special cases, like vectors of exponentials, binomial functions, logarithms and hypergeometric functions (cf. [He], [Co], [Ja], [deBr], [deBrDrLu]), a lot is known and the theory is fairly well developed, especially in the case of exponential functions.

Since in the case of only one function $f=f_1$ to be approximated, i.e. $m=1$, the definition of simultaneous rational (Hermite–Padé) approximants coincides with that of Padé approximants, it could be expected that positive results are obtainable for a vector of functions whose Padé approximants are well understood. Such a class is the set of formal power series whose Maclaurin series coefficients satisfy a certain "smoothness" condition. D. S. Lubinsky proved (cf. [Lu]) that if $f(z)=\sum_{j=0}^{\infty} a_j z^j$ and $\lim_{j\to\infty}(a_{j+1}a_{j-1})/a_j^2=q$, then with mild restrictions on the asymptotic

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behaviour of the ratio $a_{j+1}a_{j-1}/a_j^2$ in the case when q is a root of unity, all rows of the Padé table of $f(z)$ converge locally uniformly to f in $\mathbb{C}$. He also obtained (cf. [Lu]) the asymptotics of the associated Toeplitz determinants.

In this paper, we shall consider the convergence of certain sequences of simultaneous rational approximants to a pair of formal power series (f, g) where the Maclaurin series coefficients of f and g satisfy certain smoothness conditions. In order to state the results, we need to introduce some notation and a definition.

Notation. The set of all polynomials of degree $\leq n$, $n \in \mathbb{N}$, is denoted by Π_n. If $\rho_0, \rho_1, ..., \rho_m$ is an arbitrary set of non-negative integers, we call

$$\rho := (\rho_0, \rho_1, ..., \rho_m) \in \mathbb{N}^{m+1}$$

a *multi-index*, and define

$$\sigma := \rho_0 + \rho_1 + \cdots + \rho_m.$$

With this notation, we may formulate the simultaneous rational (Hermite–Padé) approximants for a pair of formal power series (f, g).

Given any $\rho = (\rho_0, \rho_1, \rho_2) \in \mathbb{N}^3$, we seek polynomials $Q \in \Pi_{\sigma-\rho_0} \backslash \{0\}$ and $P_j \in \Pi_{\sigma-\rho_j}$, $j = 1, 2$, such that

$$\begin{aligned} (fQ - P_1)(z) &= O(z^{\sigma+1}) \quad \text{as } z \to 0, \\ (gQ - P_2)(z) &= O(z^{\sigma+1}) \quad \text{as } z \to 0. \end{aligned} \tag{1.1}$$

It is well known that a non-trivial solution to (1.1) exists with $Q \not\equiv 0$.

DEFINITION 1.1. The vector $(P_1/Q, P_2/Q)$ of rational functions is called a *simultaneous rational* (or *Hermite–Padé*) *approximant* to the vector of functions (f, g).

Let $D := D(f, g; \rho_0, \rho_1, \rho_2)$ be the determinant defined by

$$D := \det \begin{bmatrix} a_{\sigma-\rho_1} & a_{\sigma-\rho_1-1} & \cdots & a_{\rho_0-\rho_1+1} \\ a_{\sigma-\rho_1+1} & a_{\sigma-\rho_1} & \cdots & a_{\rho_0-\rho_1+2} \\ \vdots & \vdots & & \vdots \\ a_{\sigma-1} & a_{\sigma-2} & \cdots & a_{\rho_0} \\ b_{\sigma-\rho_2} & b_{\sigma-\rho_2-1} & \cdots & b_{\rho_0-\rho_2+1} \\ \vdots & \vdots & & \vdots \\ b_{\sigma-1} & b_{\sigma-2} & \cdots & b_{\rho_0} \end{bmatrix}. \tag{1.2}$$

It is known (cf. [deBr]) that the approximant $(P_1/Q, P_2/Q)$ to (f, g) defined in (1.1) is unique up to normalization, provided

$$D := D(f, g; \rho_0, \rho_1, \rho_2) \neq 0. \tag{1.3}$$

Moreover, the normalized denominator polynomial $Q(z)$, with $Q(0) = 1$, is given by

$$Q(z) = \frac{1}{D} \det \begin{bmatrix} 1 & z & z^2 & \cdots & z^{\sigma-\rho_0} \\ a_{\sigma-\rho_1+1} & a_{\sigma-\rho_1} & a_{\sigma-\rho_1-1} & \cdots & a_{\rho_0-\rho_1+1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_\sigma & a_{\sigma-1} & a_{\sigma-2} & \cdots & a_{\rho_0} \\ b_{\sigma-\rho_2+1} & b_{\sigma-\rho_2} & b_{\sigma-\rho_2-1} & \cdots & b_{\rho_0-\rho_2+1} \\ \vdots & \vdots & \vdots & & \vdots \\ b_\sigma & b_{\sigma-1} & b_{\sigma-2} & \cdots & b_{\rho_0} \end{bmatrix}. \tag{1.4}$$

For an introduction to Hermite–Padé approximation, see [ApSt], [deBr].

In order to derive the maximum benefit from the identities associated with the determinants defined in (1.2) and (1.4), it becomes necessary to introduce a fourth parameter in the description of these determinants. Letting

$$r := \rho_1; \qquad s := \rho_2; \qquad l := \sigma - \rho_1; \qquad l + k := \sigma - \rho_2, \tag{1.5}$$

the determinant $D = D(\rho_0, \rho_1, \rho_2) = D(l, l+k; r, s)$ defined by (1.2) becomes

$$D = \det \begin{bmatrix} a_l & a_{l-1} & \cdots & a_{l-r-s+1} \\ a_{l+1} & a_l & \cdots & a_{l-r-s+2} \\ \vdots & \vdots & & \vdots \\ a_{l+r-1} & a_{l+r-2} & \cdots & a_{l-s} \\ b_{l+k} & b_{l+k-1} & \cdots & b_{l+k-r-s+1} \\ \vdots & \vdots & & \vdots \\ b_{l+k+s-1} & b_{l+k+s-2} & \cdots & b_{l+k-r} \end{bmatrix}. \tag{1.6}$$

This determinant is well defined for $l, r, s \in \mathbb{N}$, l sufficiently large, and $k \in \mathbb{Z}$. In the Hermite–Padé case, of course, there are only three independent parameters, namely l, r, and s. It is easily seen from (1.5) that for Hermite–Padé,

$$\rho_0 = l - s \qquad \text{and} \qquad k = r - s. \tag{1.7}$$

We observe that $k=r-s$ is the number of rows of "a" coefficients minus the number of rows of "b" coefficients. With the parameters introduced in (1.5), the normalized denominator polynomial $Q(z)=Q(l, l+k; r, s; z)$ given by (1.4) can be written

$$Q(z)=\frac{1}{D}\det\begin{bmatrix} 1 & z & z^2 & \cdots & z^{r+s} \\ a_{l+1} & a_l & a_{l-1} & \cdots & a_{l-r-s+1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{l+r} & a_{l+r-1} & a_{l+r-2} & \cdots & a_{l-s} \\ b_{l+k+1} & b_{l+k} & b_{l+k-1} & \cdots & b_{l+k-r-s+1} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{l+k+s} & b_{l+k+s-1} & b_{l+k+s-2} & \cdots & b_{l+k-r} \end{bmatrix}. \tag{1.8}$$

We now state our first main result.

THEOREM 1.1. *Let* $f(z):=\sum_{j=0}^{\infty} a_j z^j$ and $g(z):=\sum_{j=0}^{\infty} b_j z^j$ *be formal power series with* $a_j b_j \neq 0$ *for* j *large enough. Assume that*

$$\lim_{j\to\infty} \frac{a_{j+1}a_{j-1}}{a_j^2}=q, \tag{1.9}$$

and

$$\lim_{j\to\infty} \left(\frac{b_j}{b_{j+1}}\right)\Big/\left(\frac{a_j}{a_{j+1}}\right)=\lambda. \tag{1.10}$$

Suppose, in addition, that $\lambda q \neq 0$, q *is not a root of unity and* $\lambda \neq 1$.

(a) *For* $l, r, s \in \mathbb{N}$, $k \in \mathbb{Z}$ *and* $D=D(l, l+k; r, s)$ *defined by* (1.6), *we have*

$$\lim_{l\to\infty} \frac{D(l, l+k; r, s)}{a_l^r b_{l+k-r}^s} = \prod_{j=1}^{r-1}(1-q^j)^{r-j} \prod_{j=1}^{s-1}(1-q^j)^{s-j} \prod_{j=1}^{r}\prod_{n=1}^{s}(1-\lambda^{-1}q^{k+n-j}). \tag{1.11}$$

(b) *For any non-negative integer* n, *let* $B_n(u)$ *be the polynomial defined by the recurrence relation*

$$B_0(u):=1, \qquad B_n(u):=B_{n-1}(u)-uq^{n-1}B_{n-1}(uq^{-1}), \qquad n=1, 2, ... \tag{1.12}$$

Let $r, s \geqslant 0$ be integers and k any fixed integer. Define the polynomial $W_{r,s,k}(u)$ by

$$W_{r,0,k}(u) := B_r(u), \qquad r \geqslant 0, \quad \text{any fixed } k, \tag{1.13}$$

$$W_{r,s,k}(u) = W_{r,s-1,k}(u) - u\lambda^{-1} q^{k+s-1} W_{r,s-1,k}(uq^{-1}), \qquad r \geqslant 0, \quad s \geqslant 1. \tag{1.14}$$

Then

$$W_{0,s,k}(u) = B_s(uq^k \lambda^{-1}), \qquad s \geqslant 0, \quad k \text{ fixed}. \tag{1.15}$$

Further, if $Q(z) = Q(l, l+k; r, s; z)$ is the polynomial defined by (1.8), we have

$$\lim_{l \to \infty} Q(l, l+k; r, s; ua_l/a_{l+1}) = W_{r,s,k}(u), \tag{1.16}$$

locally uniformly in $\mathbb{C}$.

Remarks. (1) Theorem 1.1 is a generalization of the result proved by D. S. Lubinsky (see [Lu, Th. 1.1, p. 308]) for a formal power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$, where $\lim_{j \to \infty} (a_{j+1} a_{j-1})/a_j^2 = q$, and q is not a root of unity. Specifically, putting $s = 0$ in Theorem 1.1 yields Lubinsky's results.

(2) It is known (cf. [LuSa]) that $B_n(-u)$ is a Rogers-Szegö polynomial. When q is not a root of unity, it has been shown (see [LuSa]) that

$$B_n(-u) = \sum_{j=0}^{n} \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n+1-j})}{(1-q)(1-q^2) \cdots (1-q^j)} u^j.$$

(3) It has been mentioned that the polynomial $Q(l, l+k; r, s; z)$ is the normalized denominator polynomial in the Hermite–Padé approximant for (f, g) corresponding to the multi-index $(l-s, r, s)$ *only* when the parameter k is equal to $r-s$. It seems inappropriate therefore that k should appear as a parameter in the limit polynomials $W_{r,s,k}(u)$. However, if we study the recurrence relation (1.14), we see that when $k = r-s$, the polynomial on the right hand side of (1.14), viz. $W_{r,s-1,k}$, is not the limit polynomial of a Hermite–Padé denominator since $r-(s-1) = k+1$. A simple example, say $r=1$, $s=2$ illustrates the necessity of retaining k as an independent parameter in the recurrence relation. We have from (1.14) with $r=1$, $s=2$,

$$W_{1,2,k}(u) = W_{1,1,k}(u) - u\lambda^{-1} q^{k+1} W_{1,1,k}(uq^{-1}). \tag{1.17}$$

Then, with k fixed, $k = r - s = 1 - 2 = -1$, we see from (1.17) that in order to find $W_{1,2,-1}(u)$, we must be able to evaluate $W_{1,1,-1}(u)$ which is not the limit of the Hermite–Padé denominator for $r=1$, $s=1$ (which would be $W_{1,1,0}(u)$).

(4) We have observed that putting $s=0$ in (1.16) and using (1.13) for the Hermite–Padé case, $k=r-s=r$, we obtain

$$\lim_{l\to\infty} Q(l, l+k; r, 0; ua_l/a_{l+1}) = W_{r,0,r}(u) = B_r(u).$$

It is an immediate consequence of assumptions (1.9) and (1.10) that the Maclaurin series coefficients of $g(z)$ satisfy the smoothness condition

$$\lim_{j\to\infty} \frac{b_{j+1}b_{j-1}}{b_j^2} = \lim_{j\to\infty} \frac{a_{j+1}a_{j-1}}{a_j^2} = q.$$

It follows that for the Hermite–Padé case with $r=0$ and $k=r-s=-s$, that

$$\lim_{l\to\infty} Q(l, l-s; 0, s; ub_{l-s}/b_{l-s+1}) = B_s(u). \tag{1.18}$$

Moreover (see Lemma 2.1), we have as $l\to\infty$,

$$b_{l-s}/b_{l-s+1} = (\lambda q^s a_l/a_{l+1})(1+o(1)). \tag{1.19}$$

From (1.18) and (1.19), we deduce that

$$\lim_{l\to\infty} Q(l, l-s; 0, s; u\lambda q^s a_l/a_{l+1}) = B_s(u),$$

or

$$\lim_{l\to\infty} Q(l, l-s; 0, s; ua_l/a_{l+1}) = B_s(u\lambda^{-1}q^{-s}),$$

which is just (1.16) and (1.15) with $k=-s$.

We have remarked that with assumptions (1.9) and (1.10), the coefficients of the formal power series for f and g satisfy the same smoothness condition, which implies that f and g have the same radius of convergence. When $|q|>1$, $f(z)$ and $g(z)$ have zero radius of convergence, while when $|q|<1$, f and g are entire functions of order zero. When $|q|=1$, f and g may have zero, finite or infinite radius of convergence.

Unfortunately, the most common value of q in (1.9) is $q=1$; in this case the asymptotic (1.11) is of little use. In addition, λ defined in (1.10) may have the value 1. However, with extra conditions on the coefficients of f and g, we can extend the results of Theorem 1.1 to the case when q is a root of unity and/or $\lambda=1$.

THEOREM 1.2. *Let $f(z)=\sum_{j=0}^{\infty} a_j z^j$ and $g(z)=\sum_{j=0}^{\infty} b_j z^j$ be formal power series with $a_j b_j \neq 0$ for j large enough. Let*

$$q_l := a_{l+1} a_{l-1}/a_l^2, \tag{1.20}$$

and

$$\lambda_l := (b_l/b_{l+1})/(a_l/a_{l+1}). \tag{1.21}$$

Assume that q_l and λ_l each have an asymptotic expansion in the following sense: There exist complex numbers $\{c_k\}_{k=1}^{\infty}$ and $\{d_k\}_{k=1}^{\infty}$ with $c_1 d_1 \neq 0$ and there exist q and λ, $q\lambda \neq 0$, such that for each positive integer N,

$$q_l = q\left(1+\sum_{k=1}^{N} c_k l^{-k} + o(l^{-N})\right) \qquad as \quad l\to\infty, \tag{1.22}$$

and

$$\lambda_l = \lambda\left(1+\sum_{k=1}^{N} d_k l^{-k} + o(l^{-N})\right) \qquad as \quad l\to\infty, \tag{1.23}$$

Suppose that q is a root of unity and t is the smallest positive integer for which $q^t=1$. Then if $D=D(l, l+k; r, s)$ is defined by (1.6), we have that for $\lambda \neq 1$ or, for $\lambda = 1$,

$$d_1 \neq w t c_1 \qquad for\ any\ integer\ w, \tag{1.24}$$

then

$$\lim_{l\to\infty} \frac{D(l, l+k; r, s)}{\left(\begin{array}{c} a_l^r b_{l+k-r}^s \prod_{j=1}^{r-1}(1-q_l^j)^{r-j} \prod_{j=1}^{s-1}(1-q_l^j)^{s-j} \\ \times \prod_{j=1}^{r}\prod_{n=1}^{s}(1-\lambda_l^{-1} q_l^{k+n-j}) \end{array}\right)} = 1. \tag{1.25}$$

Furthermore, the limit relation (1.6) remains valid.

In Section 2, our main aim is the proof of Theorem 1.1. We precede this proof by the statements and proofs of two subsidiary lemmas. Section 3 contains the proof of Theorem 1.2 in addition to statements and proofs of relevant lemmas.

2. LEMMAS AND THE PROOF OF THEOREM 1.1

LEMMA 2.1. *For any positive integer l, let*

$$q_l := \frac{a_{l+1} a_{l-1}}{a_l^2} \tag{2.1}$$

and

$$\lambda_l := \frac{b_l}{b_{l+1}} \bigg/ \frac{a_l}{a_{l+1}}. \tag{2.2}$$

Then

$$\frac{a_{l+t}}{a_l} = q_l^t q_{l+1}^{t-1} \cdots q_{l+t-1} \left(\frac{a_l}{a_{l-1}}\right)^t, \qquad t > 0, \tag{2.3}$$

and

$$\frac{a_{l+t}}{a_l} = q_{l-1}^{-t-1} q_{l-2}^{-t-2} \cdots q_{l+t+1} \left(\frac{a_l}{a_{l-1}}\right)^t, \qquad t < 0. \tag{2.4}$$

Further,

$$\frac{b_{l+t}}{b_l} = \lambda_{l+t-1}^{-1} \lambda_{l+t-2}^{-1} \cdots \lambda_l^{-1} \left(\frac{a_{l+t}}{a_l}\right), \qquad t > 0, \tag{2.5}$$

and

$$\frac{b_{l+t}}{b_l} = \lambda_{l+t} \lambda_{l+t+1} \cdots \lambda_{l-1} \left(\frac{a_{l+t}}{a_l}\right), \qquad t < 0. \tag{2.6}$$

If, in addition, we have

$$\lim_{l \to \infty} q_l = q \tag{2.7}$$

and

$$\lim_{l \to \infty} \lambda_l = \lambda, \tag{2.8}$$

then for any integer t,

$$\lim_{l \to \infty} \left(\frac{a_{l+t}}{a_l}\right) \bigg/ \left(\frac{a_l}{a_{l-1}}\right)^t = q^{t(t+1)/2}, \tag{2.9}$$

and

$$\lim_{l \to \infty} \left(\frac{b_{l+t}}{b_l}\right) \bigg/ \left(\frac{a_{l+t}}{a_l}\right) = \lambda^{-t}. \tag{2.10}$$

Proof. For any positive integers l and t,

$$a_{l+t}/a_l = \prod_{m=0}^{t-1} a_{l+m+1}/a_{l+m}. \tag{2.11}$$

Now, from (2.1), for $m \geqslant 0$,

$$\begin{aligned} a_{l+m+1}/a_{l+m} &= q_{l+m}a_{l+m}/a_{l+m-1} \\ &= q_{l+m}q_{l+m-1}a_{l+m-1}/a_{l+m-2} \\ &= \cdots = q_{l+m}q_{l+m-1}\cdots q_l a_l/a_{l-1}. \end{aligned} \tag{2.12}$$

It follows from (2.11) and (2.12) that for $t > 0$,

$$a_{l+t}/a_l = q_l^t q_{l+1}^{t-1} \cdots q_{l+t-1}(a_l/a_{l-1})^t,$$

which proves (2.3). Next, for $l > 0$ and $t < 0$, we have

$$a_{l+t}/a_l = \prod_{m=1}^{-t} a_{l-m}/a_{l-m+1}. \tag{2.13}$$

From (2.1), for $m \geqslant 1$,

$$\begin{aligned} a_{l-m}/a_{l-m+1} &= q_{l-m+1}a_{l-m+1}/a_{l-m+2} \\ &= \cdots = q_{l-m+1}q_{l-m+2}\cdots q_{l-1}a_{l-1}/a_l. \end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we deduce that for $l > 0$, $t < 0$,

$$a_{l+t}/a_l = q_{l-1}^{-t-1} q_{l-2}^{-t-2} \cdots q_{l+t+1}(a_{l-1}/a_l)^{-t},$$

which establishes (2.4). The limit (2.9) follows immediately from (2.7) and (2.3) for $t > 0$, while for $t < 0$, putting $t = -s$ so that $s > 0$, we have from (2.4) that

$$a_{l+t}/a_l = a_{l-s}/a_l = q_{l-1}^{s-1} q_{l-2}^{s-2} \cdots q_{l-s+1}(a_l/a_{l-1})^{-s}.$$

Therefore, using (2.7), we obtain, with $t = -s$,

$$\begin{aligned} \lim_{l \to \infty} \frac{a_{l+t}}{a_l} \Big/ \left(\frac{a_l}{a_{l-1}}\right)^t &= q^{1+\cdots+(s-1)} = q^{s(s-1)/2} \\ &= q^{t(t+1)/2}. \end{aligned}$$

Hence (2.9) holds for all integers t, $t \neq 0$, while for $t = 0$, the statement is trivially true. To prove (2.5), we observe that for $l, t > 0$,

$$b_{l+t}/b_l = \prod_{m=1}^{t} b_{l+m}/b_{l+m-1}, \tag{2.15}$$

while, from (2.2), for $m \geqslant 1$,

$$b_{l+m}/b_{l+m-1} = \lambda_{l+m-1}^{-1} a_{l+m}/a_{l+m-1}. \tag{2.16}$$

It follows from (2.15) and (2.16) that for $l, t > 0$,

$$\begin{aligned} b_{l+t}/b_l &= \lambda_l^{-1}\lambda_{l+1}^{-1}\cdots\lambda_{l+t-1}^{-1} \prod_{m=1}^{t} a_{l+m}/a_{l+m-1} \\ &= \lambda_l^{-1}\lambda_{l+1}^{-1}\cdots\lambda_{l+t-1}^{-1} a_{l+t}/a_l. \end{aligned}$$

This proves (2.5). Further, for $l > 0$ and $t < 0$, we have

$$b_{l+t}/b_l = \prod_{m=1}^{-t} b_{l-m}/b_{l-m+1}. \tag{2.17}$$

Also, from (2.2), for $m \geqslant 1$,

$$b_{l-m}/b_{l-m+1} = \lambda_{l-m} a_{l-m}/a_{l-m+1}. \tag{2.18}$$

Therefore, from (2.17) and (2.18), for $l > 0$, $t < 0$,

$$\begin{aligned} b_{l+t}/b_l &= \prod_{m=1}^{-t} \lambda_{l-m} a_{l-m}/a_{l-m+1} \\ &= \lambda_{l+t}\lambda_{l+t+1}\cdots\lambda_{l-1} a_{l+t}/a_l, \end{aligned}$$

and we have shown that (2.6) holds. The limit (2.10) follows from (2.5), (2.6), and (2.7). ∎

Lemma 2.2. *Let l, r and s be positive integers, $l \geqslant s$, and k any integer. Let $D = D(l, l+k; r, s)$ be defined for $r+s > 0$ by*

$$D(l, l+k; r, s) := \det \begin{bmatrix} a_l & a_{l-1} & \cdots & a_{l-r-s+1} \\ \vdots & \vdots & & \vdots \\ a_{l+r-1} & a_{l+r-2} & \cdots & a_{l-s} \\ b_{l+k} & b_{l+k-1} & \cdots & b_{l+k-r-s+1} \\ \vdots & \vdots & & \vdots \\ b_{l+k+s-1} & b_{l+k+s-2} & \cdots & b_{l+k-r} \end{bmatrix}. \tag{2.19}$$

while

$$D(l, l+k; 0, 0) := 1. \tag{2.20}$$

We define

$$D(l, l+k; r, -1) := D(l, l+k; r-1, 0), \qquad r \geqslant 1, \tag{2.21}$$

$$D(l, l+k; -1, s) := D(l, l+k+1; 0, s-1), \qquad s \geqslant 1. \tag{2.22}$$

Then, for $r, s \geqslant 0$, $r+s > 1$, *we have*

$$\begin{aligned} D(l, l+k; r, s)\, D(l, l+k-1; r-1, s-1) &= D(l, l+k-1; r-1, s)\, D(l, l+k; r, s-1) \\ &\quad - D(l+1, l+k; r-1, s)\, D(l-1, l+k-1; r, s-1). \end{aligned} \tag{2.23}$$

Further, if $Q(l, l+k; r, s; z)$ *is the polynomial defined by* (1.8) *and* $D(l, l+k; r, s)$, $D(l, l+k; r, s-1)$ *and* $D(l-1, l+k-1; r, s-1)$ *are non-zero, then*

$$Q(l, l+k; r, s; z) = Q(l, l+k; r, s-1; z) - zQ(l-1, l+k-1; r, s-1; z)\, X, \tag{2.24}$$

where

$$X := \frac{D(l-1, l+k-1; r, s-1)\, D(l+1, l+k+1; r, s)}{D(l, l+k; r, s-1)\, D(l, l+k; r, s)}. \tag{2.25}$$

Proof. For $r \geqslant 1$, $s \geqslant 1$, (2.23) follows from a special case of Sylvester's identity (cf. [BaGr, p. 23]) which states the following: Let C be a $k \times k$ matrix and let $1 \leqslant p, q, m, n \leqslant k$. Let $C_{m;p}(C_{m,n;p,q})$ denote the matrix obtained from C by deleting the mth row and pth column (respectively, the mth and nth rows and pth and qth columns). Then

$$(\det C)(\det C_{1,k;1,k}) = (\det C_{1;1})(\det C_{k;k}) - (\det C_{1,k})(\det C_{k;1}). \tag{2.26}$$

Applying the identity (2.26) to the $(r+s) \times (r+s)$ matrix of which $D(l, l+k; r, s)$ is the determinant, we obtain (2.23). It remains to show that (2.23) holds for $r = 0$, $s \geqslant 2$ and for $r \geqslant 2$, $s = 0$. First, for $r = 0$, $s \geqslant 2$, the left hand side of (2.23) is

$$\begin{aligned} &D(l, l+k; 0, s)\, D(l, l+k-1; -1, s-1) \\ &\quad = D(l, l+k; 0, s)\, D(l, l+k; 0, s-2), \end{aligned}$$

by definition (2.22). The right hand side of (2.23) is

$$\begin{aligned}
&D(l, l+k; -1, s)\, D(l, l+k; 0, s-1) \\
&\qquad - D(l+1, l+k; -1, s)\, D(l-1, l+k-1; 0, s-1) \\
&\quad = D(l, l+k; 0, s-1)\, D(l, l+k; 0, s-1) \\
&\qquad - D(l+1, l+k+1; 0, s-1)\, D(l-1, l+k-1; 0, s-1),
\end{aligned}$$

by (2.22). Therefore, for $r=0$, $s \geqslant 2$, (2.23) holds since it is just the identity (2.26) applied to the matrix of which $D(l, l+k; 0, s)$ is the determinant. Similarly, using (2.21), we can check that (2.23) holds for $r \geqslant 2$, $s=0$.

In order to prove (2.24), let $V(z) = V(l, l+k; r, s; z)$ denote the polynomial defined by

$$V(l, l+k; r, s; z)$$

$$:= \det \begin{bmatrix}
1 & z & z^2 & \cdots & z^{r+s} \\
a_{l+1} & a_l & a_{l-1} & \cdots & a_{l-r-s+1} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{l+r} & a_{l+r-1} & a_{l+r-2} & \cdots & a_{l-s} \\
b_{l+k+1} & b_{l+k} & b_{l+k-1} & \cdots & b_{l+k-r-s+1} \\
\vdots & \vdots & \vdots & & \vdots \\
b_{l+k+s} & b_{l+k+s-1} & b_{l+k+s-2} & \cdots & b_{l+k-r}
\end{bmatrix}. \tag{2.27}$$

Clearly, from (1.8) and (2.27), we have

$$V(l, l+k; r, s; z) = D(l, l+k; r, s)\, Q(l, l+k; r, s; z). \tag{2.28}$$

Applying the identity (2.26) to (2.27), we obtain for $r \geqslant 0$, $s \geqslant 1$,

$$\begin{aligned}
&V(l, l+k; r, s; z)\, D(l, l+k; r, s-1) \\
&\quad = V(l, l+k; r, s-1; z)\, D(l, l+k; r, s) \\
&\qquad - zV(l-1, l+k-1; r, s-1; z)\, D(l+1, l+k+1; r, s).
\end{aligned} \tag{2.29}$$

Dividing (2.29) by $D(l, l+k; r, s)\, D(l, l+k; r, s-1)$ and using (2.28) yields (2.24) and (2.25). ∎

We shall now prove Theorem 1.1 using induction on $(r+s)$, $r, s \in \mathbb{N}$ and the recursion relation (2.23). An alternative proof of (a) that avoids induction and recursion has been pointed out by one of the referees. In brief outline, if one factors the first element of each row of D in (1.6), followed by a suitable factoring of each column, one can show that D is asymptotically equal to some factors times the Vandermonde determinant of 1, $1/q$, $1/q^2, \ldots, 1/q^{r-1}, \lambda/q^k, \lambda/q^{k+1}, \ldots, \lambda/q^{k+s-1}$ and (1.11) can be deduced.

Proof of Theorem 1.1. (a) For $r=0$, $s=0$, (1.11) holds since $D(l, l+k; 0, 0)=1$. Further, from (1.6), we have

$$D(l, l+k; 1, 0)=a_l \qquad \text{and} \qquad D(l, l+k; 0, 1)=b_{l+k},$$

so that (1.11) is true for $r+s \leqslant 1$. Now assume that (1.11) holds for all non-negative integers r and s with $r+s \leqslant m$. We shall prove that (1.11) is true for $(r+1, s)$ and $(r, s+1)$ where $r+s=m$. From (2.23), we have

$$\begin{aligned} &D(l, l+k; r+1, s)\, D(l, l+k-1; r, s-1) \\ &\quad = D(l, l+k-1; r, s)\, D(l, l+k; r+1, s-1) \\ &\qquad - D(l+1, l+k; r, s)\, D(l-1, l+k-1; r+1, s-1). \end{aligned} \tag{2.30}$$

Applying the inductive hypothesis to the determinants of the matrices of order $\leqslant m$ in (2.30), we obtain that for q not a root of unity, $\lambda \neq 1$, $\lambda q \neq 0$, as $l \to \infty$

$$\begin{aligned} D(l, l+k; r+1, s) &= a_l^{r+1} b_{l+k-r-1}^{s} \prod_{j=1}^{s-1} (1-q^j)^{s-j} \prod_{j=1}^{r} (1-q^j)^{r+1-j} \\ &\quad \times \prod_{j=1}^{r} (1-\lambda^{-1} q^{k+s-1-j}) \prod_{j=1}^{r+1} \prod_{n=1}^{s-1} (1-\lambda^{-1} q^{k+n-j}) \\ &\quad \times (1-Y)(1+o(1)), \end{aligned} \tag{2.31}$$

where

$$Y := \left(\frac{a_{l+1}}{a_l}\right)^r \left(\frac{b_{l+k-r}}{b_{l+k-r-1}}\right)^s \left(\frac{a_{l-1}}{a_l}\right)^{r+1} \left(\frac{b_{l+k-r-2}}{b_{l+k-r-1}}\right)^{s-1}. \tag{2.32}$$

From (2.32), (2.9), and (2.10), we have as $l \to \infty$

$$\begin{aligned} Y &= q^r \left(\frac{a_l}{a_{l-1}}\right)^r \left(\frac{q^{k-r}}{\lambda} \frac{a_l}{a_{l-1}}\right)^s \left(\frac{a_{l-1}}{a_l}\right)^{r+1} \\ &\quad \times \left(\frac{\lambda}{q^{k-r-1}} \frac{a_{l-1}}{a_l}\right)^{s-1} (1+o(1)) \\ &= \lambda^{-1} q^{k+s-1} (1+o(1)). \end{aligned} \tag{2.33}$$

Therefore, from (2.33), as $l\to\infty$,

$$\begin{aligned}\prod_{j=1}^{r}(1-\lambda^{-1}q^{k+s-1-j})\prod_{j=1}^{r+1}\prod_{n=1}^{s-1}(1-\lambda^{-1}q^{k+n-j})(1-Y)\\ =\prod_{j=1}^{r+1}(1-\lambda^{-1}q^{k+s-j})\prod_{j=1}^{r+1}\prod_{n=1}^{s-1}(1-\lambda^{-1}q^{k+n-j})(1+o(1))\\ =\prod_{j=1}^{r+1}\prod_{n=1}^{s}(1-\lambda^{-1}q^{k+n-j})(1+o(1)).\end{aligned}\tag{2.34}$$

Substituting (2.34) into (2.31), we have as $l\to\infty$,

$$\begin{aligned}D(l,l+k;r+1,s)=a_l^{r+1}b_{l+k-r-1}^{s}\prod_{j=1}^{r}(1-q^j)^{r+1-j}\prod_{j=1}^{s-1}(1-q^j)^{s-j}\\ \times\prod_{j=1}^{r+1}\prod_{n=1}^{s}(1-\lambda^{-1}q^{k+n-j})(1+o(1)),\end{aligned}$$

which establishes (1.11) for $(r+1,s)$. A similar calculation yields (1.11) for $(r,s+1)$. Therefore, by induction, (1.11) holds for all non-negative integers r and s, and we have completed the proof of (a).

(b) We again use induction to prove (1.15). First, from (1.13) with $r=0$, we have (with $s=0$),

$$W_{0,0,k}(u)=B_0(u)=1,\tag{2.35}$$

by (1.12), so that (1.15) holds for $s=0$. Also, from (1.14), for any fixed integer k and $r=0$, we have (with $s=1$)

$$W_{0,1,k}(u)=W_{0,0,k}(u)-u\lambda^{-1}q^kW_{0,0,k}(uq^{-1})=1-u\lambda^{-1}q^k,\tag{2.36}$$

by (2.35). Further, from (1.12),

$$B_1(u\lambda^{-1}q^k)=1-u\lambda^{-1}q^k,\tag{2.37}$$

so that (2.36) and (2.37) establish (1.15) for $s=1$. Assume as an inductive hypothesis that (1.15) holds for all integers $s\leqslant m-1$, $m\in\mathbb{N}$. Then by (1.14) with $s=m$,

$$\begin{aligned}W_{0,m,k}(u)&=W_{0,m-1,k}(u)-u\lambda^{-1}q^{k+m-1}W_{0,m-1,k}(uq^{-1})\\ &=B_{m-1}(u\lambda^{-1}q^k)-u\lambda^{-1}q^kq^{m-1}B_{m-1}(u\lambda^{-1}q^kq^{-1})\\ &=B_m(u\lambda^{-1}q^k),\end{aligned}$$

where we have used the inductive hypothesis and (1.12). Therefore (1.15) holds for $s=m$ and by induction, we have proved (1.15) for all $s \geqslant 0$ and any fixed $k \in \mathbb{Z}$.

In order to prove (1.16), we use (2.24) and (2.25) together with (1.11). From (2.25) and (1.11) we have as $l \to \infty$,

$$X=\left(\frac{a_{l-1}a_{l+1}}{a_l^2}\right)^r \left(\frac{b_{l+k-r-1}b_{l+k-r+1}}{b_{l+k-r}^2}\right)^{s-1} \left(\frac{b_{l+k-r+1}}{b_{l+k-r}}\right)(1+o(1)). \tag{2.38}$$

Now, from (2.9) and (2.10), for any fixed integers k and r,

$$\lim_{l\to\infty} \frac{b_{l+k-r-1}b_{l+k-r+1}}{b_{l+k-r}^2} = \lim_{l\to\infty} \frac{a_{l-1}a_{l+1}}{a_l^2} = q, \tag{2.39}$$

and

$$\lim_{l\to\infty} \frac{b_{l+k-r+1}}{b_{l+k-r}} = \lambda^{-1} q^{k-r+1} \lim_{l\to\infty} \frac{a_l}{a_{l-1}}. \tag{2.40}$$

Therefore, from (2.38), (2.39), and (2.40), we obtain as $l \to \infty$,

$$\begin{aligned} X &= q^r q^{s-1} \lambda^{-1} q^{k-r+1}(a_l/a_{l-1})(1+o(1)) \\ &= \lambda^{-1} q^{k+s}(a_l/a_{l-1})(1+o(1)). \end{aligned} \tag{2.41}$$

Substituting for X from (2.41) into (2.24) yields as $l \to \infty$,

$$\begin{aligned} Q(l, l+k; r, s; z) &= Q(l, l+k; r, s-1; z) - z\lambda^{-1} q^{k+s} a_l/a_{l-1} \\ &\quad \times Q(l-1, l+k-1; r, s-1; z)(1+o(1)). \end{aligned} \tag{2.42}$$

Putting $z = ua_l/a_{l+1}$ in (2.42) and observing that then $za_l/a_{l-1} = uq_l^{-1}$, we obtain as $l \to \infty$,

$$\begin{aligned} &Q(l, l+k; r, s; ua_l/a_{l+1}) \\ &\quad = Q(l, l+k; r, s-1; ua_l/a_{l+1}) \\ &\qquad - \lambda^{-1} q^{k+s-1} u Q(l-1, l+k-1; r, s-1; q^{-1} ua_l/a_{l+1})(1+o(1)). \end{aligned} \tag{2.43}$$

Armed with (2.43), we now prove (1.16) by induction on $(r+s)$. From definition (1.8),

$$Q(l, l+k; 0, 0; ua_l/a_{l+1}) = 1. \tag{2.44}$$

Also from (1.8), with $r=1$, $s=0$, we have

$$Q(l, l+k; 1, 0; ua_l/a_{l+1}) = \frac{1}{a_l} \det \begin{bmatrix} 1 & ua_l/a_{l+1} \\ a_{l+1} & a_l \end{bmatrix}$$

$$= \frac{1}{a_l}(a_l - ua_l) = 1-u. \tag{2.45}$$

On the other hand, from (1.8) with $r=0$, $s=1$,

$$Q(l, l+k; 0, 1; ua_l/a_{l+1})$$

$$= \frac{1}{b_{l+k}} \det \begin{bmatrix} 1 & ua_l/a_{l+1} \\ b_{l+k+1} & b_{l+k} \end{bmatrix}$$

$$= 1 - u \frac{a_l}{a_{l+1}} \frac{b_{l+k+1}}{b_{l+k}}$$

$$= 1 - u\left(\frac{a_l}{a_{l+1}}\right) \lambda^{-1} q^{k+1} \left(\frac{a_l}{a_{l-1}}\right)(1+o(1)) \qquad \text{as } l \to \infty,$$

where we have used (2.9) and (2.10). Therefore,

$$\lim_{l \to \infty} Q(l, l+k; 0, 1; ua_l/a_{l+1}) = 1 - \lambda^{-1} q^k u$$

$$= W_{0,1,k}(u), \tag{2.46}$$

by (2.36). Therefore, we see from (2.44), (2.45) and (2.46) that (1.16) holds for $r, s \geqslant 0$, $r+s \leqslant 1$. Assume as an inductive hypothesis that (1.16) holds for all non-negative integers r and s with $r+s \leqslant m-1$, $m \in \mathbb{N}$. Then, with $r+s=m$, we have from (2.43) and the inductive hypothesis that with k a fixed integer, as $l \to \infty$,

$$Q(l, l+k; r, s; ua_l/a_{l+1})$$

$$= W_{r,s-1,k}(u) + o(1) - \lambda^{-1} q^{k+s-1} u W_{r,s-1,k}(uq^{-1}) + o(1)$$

$$= W_{r,s,k}(u) + o(1),$$

by the recurrence relation (1.14). Therefore, (1.16) is true for $r+s=m$ and therefore by induction, we have proved (1.16) for all non-negative integers r, s and k fixed, $k \in \mathbb{Z}$. Observe that (1.16) holds uniformly for u in compact subsets of $\mathbb{C}$, since $q\lambda \neq 0$. ∎

3. Asymptotics When q Is a Root of Unity and/or $\lambda = 1$

We introduce an analogue for asymptotic series of the O, o, notation (cf. [Lu, p. 310]). Given non-negative integers S and T with $S \leqslant T$, and given a sequence of complex numbers $\{e_l\}_{l=1}^{\infty}$, we write

$$e_l = A_l[S; T] \tag{3.1}$$

if and only if there exist $C_S, C_{S+1}, ..., C_T \in \mathbb{C}$ with

$$e_l = \sum_{k=S}^{T} C_k l^{-k} + o(l^{-T}) \qquad \text{as} \quad l \to \infty. \tag{3.2}$$

Lemma 3.1. *Let l be a positive integer and let*

$$q_l := a_{l+1} a_{l-1} / a_l^2, \tag{3.3}$$

$$\lambda_l := (b_l / b_{l+1}) / (a_l / a_{l+1}). \tag{3.4}$$

Assume that q_l and λ_l each have a complete asymptotic expansion in the sense that there exist complex numbers $\{c_k\}_{k=1}^{\infty}$, $\{d_k\}_{k=1}^{\infty}$ with

$$c_1 d_1 \neq 0, \tag{3.5}$$

and there exist q and λ, $q\lambda \neq 0$, such that for each positive integer N,

$$q_l = q\left(1 + \sum_{k=1}^{N} c_k l^{-k} + o(l^{-N})\right) \qquad as \quad l \to \infty, \tag{3.6}$$

and

$$\lambda_l = \lambda\left(1 + \sum_{k=1}^{N} d_k l^{-k} + o(l^{-N})\right) \qquad as \quad l \to \infty. \tag{3.7}$$

With the notation in (3.1) and (3.2), and N an arbitrary positive integer $\geqslant 3$, we have the following:

(a) *For any integer t,*

$$q_{l+t} = q_l(1 + A_l[2; N]) \qquad as \quad l \to \infty, \tag{3.8}$$

and

$$\lambda_{l+t} = \lambda_l(1 + A_l[2; N]) \qquad as \quad l \to \infty. \tag{3.9}$$

(b) *For any fixed positive integer j,*

$$1 - q_l^j \neq 0 \qquad \text{for } l \text{ sufficiently large}, \tag{3.10}$$

and

$$(1-q_{l\pm 1}^j)/(1-q_l^j)=1+A_l[1;N], \qquad l\to\infty. \tag{3.11}$$

(c) *For any integer t with*

$$tc_1-d_1\neq 0, \tag{3.12}$$

we have

$$1-\lambda_l^{-1}q_l^t\neq 0 \qquad \textit{for } l \textit{ sufficiently large}, \tag{3.13}$$

and

$$(1-\lambda_{l\pm 1}^{-1}q_{l\pm 1}^t)/(1-\lambda_l^{-1}q_l^t)=1+A_l[1;N], \qquad l\to\infty. \tag{3.14}$$

Proof. (a) From (3.6) we have as $l\to\infty$, $t\in\mathbb{Z}$ fixed,

$$\begin{aligned} q_{l+t}&=q\left(1+\frac{c_1}{l+t}+\frac{c_2}{(l+t)^2}+\cdots+\frac{c_N}{(l+t)^N}+o(l+t)^{-N}\right)\\ &=q\left[1+\frac{c_1}{l}\left(1-\frac{t}{l}+\frac{t^2}{l^2}-\cdots\right)+\cdots+\frac{c_N}{l^N}\left(1-\frac{Nt}{l}+\cdots\right)+o(l^{-N})\right]\\ &=q\left(1+\frac{c_1}{l}+\cdots+\frac{c_N}{l^N}+A_l[2;N]\right)\\ &=q_l(1+A_l[2;N]), \end{aligned}$$

which proves (3.8). The proof of (3.9) follows the same procedure.

(b) For any fixed positive integer j, we have by (3.6), as $l\to\infty$,

$$q_l^j=q^j\left(1+\frac{jc_1}{l}+A_l[2;N]\right). \tag{3.15}$$

Therefore,

$$1-q_l^j=1-q^j-q^jjc_1/l+A_l[2;N], \tag{3.16}$$

which is non-zero even when $q^j=1$ since $c_1\neq 0$ by (3.5). This establishes (3.10). Also, from (3.6), as $l\to\infty$,

$$q_l^j=q^j\left(1+\frac{jc_1}{l}+\frac{jc_2}{l^2}+A_l[3;N]\right),$$

$$q_{l\pm 1}^j=q^j\left(1+\frac{jc_1}{l\pm 1}+\frac{jc_2}{(l\pm 1)^2}+A_{l+1}[3;N]\right).$$

Therefore, as $l \to \infty$,

$$q_l^j - q_{l\pm 1}^j = q^j j\left[c_1\left(\frac{1}{l} - \frac{1}{l\pm 1}\right) + c_2\left(\frac{1}{l^2} + \frac{1}{(l\pm 1)^2}\right) + A_l[3; N]\right]$$

$$= \pm\left(\frac{q^j jc_1}{l^2}\right)(1 + A_l[1; N-2]). \tag{3.17}$$

Also, from (3.16),

$$1 - q_l^j = \begin{cases}(1-q^j)(1+A_l[1; N]) & \text{for } q^j \neq 1, \\ (-jc_1/l)(1+A_l[1; N-1]) & \text{for } q^j = 1.\end{cases} \tag{3.18}$$

Now

$$(1 - q_{l\pm 1}^j)/(1 - q_l^j) = 1 + (q_l^j - q_{l\pm 1}^j)/(1 - q_l^j). \tag{3.19}$$

From (3.17) and (3.18) it follows that

$$(q_l^j - q_{l\pm 1}^j)/(1 - q_l^j) = \begin{cases}A_l[2; N'], & \text{if } q^j \neq 1 \\ A_l[1; N''], & \text{if } q^j = 1,\end{cases} \tag{3.20}$$

where N', N'' depend on N. Since N is arbitrary, (3.11) follows immediately from (3.19) and (3.20).

(c) For any $t \in \mathbb{Z}$, by (3.6) and (3.7), we have as $l \to \infty$,

$$\lambda_l^{-1} q_l^t = \lambda^{-1} q^t\left(1 - \frac{d_1}{l} + A_l[2; N]\right)\left(1 + \frac{tc_1}{l} + A_l[2; N]\right)$$

$$= \lambda^{-1} q^t\left(1 + \frac{tc_1 - d_1}{l} + A_l[2; N]\right). \tag{3.21}$$

Therefore, as $l \to \infty$,

$$1 - \lambda_l^{-1} q_l^t = 1 - \lambda^{-1} q^t - \frac{\lambda^{-1} q^t(tc_1 - d_1)}{l} + A_l[2; N], \tag{3.22}$$

and (3.13) follows even when $\lambda^{-1} q^t = 1$, since $tc_1 - d_1 \neq 0$ by (3.12). Next,

$$(1 - \lambda_{l\pm 1}^{-1} q_{l\pm 1}^t)/(1 - \lambda_l^{-1} q_l^t)$$
$$= 1 + (\lambda_l^{-1} q_l^t - \lambda_{l\pm 1}^{-1} q_{l\pm 1}^t)/(1 - \lambda_l^{-1} q_l^t). \tag{3.23}$$

From (3.21), as $l \to \infty$,

$$\lambda_l^{-1} q_l' - \lambda_{l\pm 1}^{-1} q_{l\pm 1}' = \lambda^{-1} q' \left[d_1 \left(\frac{1}{l\pm 1} - \frac{1}{l} \right) + tc_1 \left(\frac{1}{l} - \frac{1}{l\pm 1} \right) + A_l[3; N] \right]$$

$$= \left(\frac{\pm \lambda^{-1} q'(tc_1 - d_1)}{l^2} \right) (1 + A_l[1; N]). \qquad (3.24)$$

Also, from (3.22), as $l \to \infty$,

$$1 - \lambda_l^{-1} q_l' = \begin{cases} (1 - \lambda^{-1} q')(1 + A_l[1; N]) & \text{for } \lambda^{-1} q' \neq 1, \\ -\dfrac{(tc_1 - d_1)}{l} (1 + A_l[1; N]) & \text{for } \lambda^{-1} q' = 1. \end{cases} \qquad (3.25)$$

From (3.24) and (3.25) we deduce that as $l \to \infty$,

$$\frac{(\lambda_l^{-1} q_l' - \lambda_{l\pm 1}^{-1} q_{l\pm 1}')}{(1 - \lambda_l^{-1} q_l')} = \begin{cases} A_l[2; N] & \text{if } \lambda^{-1} q' \neq 1, \\ A_l[1; N] & \text{if } \lambda^{-1} q' = 1, \end{cases} \qquad (3.26)$$

and (3.14) follows from (3.23) and (3.26).

Proof of Theorem 1.2. We use induction on $(r+s)$, recalling that r and s are non-negative integers. From the definition (1.6) of $D(l, l+k; r, s)$, we see that $D(l, l+k; 0, 0) = 1$; $D(l, l+k; 1, 0) = a_l$; $D(l, l+k; 0, 1) = b_{l+k}$. Therefore, (1.25) holds for all non-negative integers r and s with $r+s \leqslant 1$. Assume as an inductive hypothesis that for all non-negative integers r and s with $r+s \leqslant m-1$, $m > 1$, and for an arbitrary $N \in \mathbb{N}$, as $l \to \infty$,

$$D(l, l+k; r, s) = a_l^r b_{l+k-r}^s \prod_{j=1}^{r-1} (1 - q_l^j)^{r-j} \prod_{j=1}^{s-1} (1 - q_l^j)^{s-j}$$

$$\times \prod_{j=1}^{r} \prod_{n=1}^{s} (1 - \lambda_l^{-1} q_l^{k+n-j})(1 + A_l[1; N]). \qquad (3.27)$$

Suppose now that $r+s = m-1$. Using the identity (2.23) together with the inductive hypothesis (3.26), a straightforward calculation yields, as $l \to \infty$,

$$D(l, l+k; r+1, s) = a_l^{r+1} b_{l+k-r-1}^s \prod_{j=1}^{r} (1 - q_l^j)^{r+1-j} \prod_{j=1}^{s-1} (1 - q_l^j)^{s-j}$$

$$\times \prod_{j=1}^{r} (1 - \lambda_l^{-1} q_l^{k+s-j-1}) \prod_{j=1}^{r+1} \prod_{n=1}^{s-1} (1 - \lambda_l^{-1} q_l^{k+n-j})$$

$$\times (1 - Z)(1 + A_l[1; N]), \qquad (3.28)$$

where

$$Z=\left(\frac{a_{l+1}}{a_l}\right)^r\left(\frac{a_{l-1}}{a_l}\right)^{r+1}\left(\frac{b_{l+k-r-2}}{b_{l+k-r-1}}\right)^{s-1}\left(\frac{b_{l+k-r}}{b_{l+k-r-1}}\right)^s$$
$$\times\prod_{j=1}^{r-1}\left(\frac{1-q_{l+1}^j}{1-q_l^j}\right)^{r-j}\prod_{j=1}^{s-1}\left(\frac{1-q_{l+1}^j}{1-q_l^j}\right)^{s-j}$$
$$\times\prod_{j=1}^{r}\prod_{n=1}^{s-1}\left(\frac{1-\lambda_{l+1}^{-1}q_{l+1}^{k+n-j-1}}{1-\lambda_l^{-1}q_l^{k+n-j-1}}\right)\prod_{j=1}^{r}\left(\frac{1-q_{l-1}^j}{1-q_l^j}\right)^{r+1-j}$$
$$\times\prod_{j=1}^{r}\left(\frac{1-\lambda_{l+1}^{-1}q_{l+1}^{k+s-j-1}}{1-\lambda_l^{-1}q_l^{k+s-j-1}}\right)\prod_{j=1}^{s-2}\left(\frac{1-q_{l-1}^j}{1-q_l^j}\right)^{s-1-j}$$
$$\times\prod_{j=1}^{r+1}\prod_{n=1}^{s-1}\left(\frac{1-\lambda_{l-1}^{-1}q_{l-1}^{k+n-j}}{1-\lambda_l^{-1}q_l^{k+n-j}}\right). \tag{3.29}$$

Observe that the division evident in (3.28) and (3.29) is justified because of (3.10) and (3.13). Now, by (2.3), (2.4), (2.5), and (2.6), we have

$$\left(\frac{a_{l+1}}{a_l}\right)^r\left(\frac{a_{l-1}}{a_l}\right)^{r+1}\left(\frac{b_{l+k-r-2}}{b_{l+k-r-1}}\right)^{s-1}\left(\frac{b_{l+k-r}}{b_{l+k-r-s}}\right)^s$$
$$=q_l^r q_{l+k-r-1}^s q_l\cdots q_{l+k-r-2}\frac{\lambda_{l+k-r-s}}{\lambda_{l+k-r-1}}\lambda_{l+k-r-1}^{-1}$$
$$=\lambda_l^{-1}q_l^{k+s-1}[1+A_l[2;N]], \tag{3.30}$$

where we have used (3.8), (3.9) and the fact that $(1+A_l[2;N])'=1+A_l[2;N]$. Furthermore, each of the remaining terms in the product on the right hand side of (3.29) is of the form $1+A_l[1;N]$, by (3.11) and (3.14). Then from (3.29), (3.30) and the previous remark, we deduce that

$$Z=\lambda_l^{-1}q_l^{k+s-1}(1+A_l[1;N]) \qquad \text{as} \quad l\to\infty. \tag{3.31}$$

It follows from (3.28) and (3.31) that as $l\to\infty$,

$$D(l,l+k;r+1,s)=a_l^{r+1}b_{l+k-r-1}^s\prod_{j=1}^{r}(1-q_l^j)^{r+1-j}\prod_{j=1}^{s-1}(1-q_l^j)^{s-j}$$
$$\times\prod_{j=1}^{r+1}\prod_{n=1}^{s}(1-\lambda_l^{-1}q_l^{k+n-j})(1+A_l[1;N]). \tag{3.32}$$

A similar calculation shows that $D(l,l+k;r,s+1)$ has a complete asymptotic expansion of the required form and therefore by induction we have proved (1.25) for all $r,s\geqslant 0$ and $k\in\mathbb{Z}$. Finally, the limit relation (1.16) follows from (1.25) and the proof of (1.16) in Theorem 1.1(b). ∎

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